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Note on the paper of Kochar & Jain on Howard's semicircle theorem

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A further refinement of the Howard–Kochar–Jain theorem is given which allows the estimation of the range of complex wave velocity for growing perturbations in a stratified shear flow. According to the results obtained, the boundary of this region depends both on the minimum Richardson number and on the wavenumber of the perturbations. The effect of external boundaries on the stability of parallel flows is defined. An estimate of the maximum rate of growth versus dimensionless wavenumber is found. The theoretical results are compared with numerical computations and laboratory experiments of other authors.

1. Introduction

Stability of stratified shear flows in an inviscid incompressible fluid of variable density is of considerable theoretical and practical interest for hydrodynamics, oceanography, geophysics, physics of atmospheres, etc. Calculation of eigenvalues of the singular Taylor–Goldstein equation generalizing the well-known Rayleigh equation for the case of a stratified fluid presents certain difficulties in solving this problem. Only a few examples of shear flows with a simple configuration are available when the Taylor–Goldstein equation is solved analytically (Drazin & Howard 1966; Turner 1973; Gossard & Hooke 1975). Numerical solutions are most commonly used, but they are also rather laborious (Hazel 1972). In this situation, various integral estimates acquire great importance, for they enable one to obtain sufficient conditions of stability or instability (Yih 1974) and define a possible range of parameters for growing perturbations in the case of instability. Important results in this direction, widely used now by oceanographers and specialists in atmospheric physics, have been obtained by Miles (1961) and Howard (1961). Later, Howard's semicircle theorem, according to which the complex wave velocity for any unstable mode lies inside the semicircle defined by the maximum and minimum velocities of the main flow, was refined by Kochar & Jain (1979). According to their results, the complex wave velocity for unstable modes lies in a semi-ellipse whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends on the minimum Richardson number $J_0 = \min\,[N(y)/U'(y)]^2$, where $U(y)$ is the velocity of the parallel flow oriented along the x-axis, $N(y) = [g\beta]^{\frac{1}{2}}$ is the Brunt–Väisälä frequency, g is the acceleration due to gravity, and $\beta = -\rho'(y)/\rho(y)$ is the logarithmic derivative of the density profile. However, neither Howard's theorem nor its generalization by Kochar & Jain take into account the dependence of the size of the instability region on the perturbation wavenumber. At the same time, it is known that this dependence is quite substantial (Drazin & Howard 1966; Turner 1973; Gossard & Hooke 1975). For example, even in the 'most-unstable' case $J_0 = 0$, only perturbations with wavenumbers $k \lesssim 1/h$ can grow (h is the characteristic thickness of the shear layer) and perturbations with $\alpha \equiv kh \approx 0.5$ have the maximum rate of growth. Howard (1961) gave the upper limit for the imaginary part of the wave velocity, which takes into account dependence on both the wavenumber and the flow parameters. This estimate, however, does not enter into the theorem, but only supplements it.

This paper gives a further refinement of the Howard–Kochar–Jain theorem. A basic inequality for arbitrary velocity and density profiles is derived in §2. This inequality generalizes results of the previous investigations (Howard 1961; Kochar & Jain 1979) and comprises parameters such as the Richardson number J_0 and the wavenumber k of the unstable mode. This enables one to narrow the possible range of phase velocity of unstable modes, as well as to obtain the estimates of the maximum rate of growth kC_i versus J_0 and k. In §3 a generalization of Howard's (1961) inequality is given for estimation of the maximum kC_i. The obtained relation (12) takes into account the total thickness of the fluid layer with velocity shear. Numerical computations (Hazel 1972) show that in the presence of external boundaries the instability region in the (J_0, α) plane may change essentially, which agrees with our formula (12). Section 4 deals with some particular flow profiles, among which is a hyperbolic-tangent profile frequently used in practical calculations. For these profiles we refine the formerly obtained boundaries of instability regions. Section 5 presents comparison of our results with the results obtained by Howard (1961) and Kochar & Jain (1979), as well as correlation of the estimate of the rate of growth versus wavenumber with the numerical computations (Hazel 1972) and the laboratory data (Scotti & Corcos 1969).

2. Basic results for arbitrary velocity and density profiles

The basic equation describing small perturbations in a stratified shear flow (the Taylor–Goldstein equation) can be readily obtained from the linearized Euler and continuity equations, assuming that all the variables are proportional to $\exp[ik(x-ct)]$. For vertical deviation of the constant-density line from the mean position $\eta(x,y,t) = F(y)\exp[ik(x-ct)]$ it has the form (Miles 1961)

$$[\rho(U-c)^2 F']' + \rho[N^2(y) - k^2(U-c)^2]F = 0. \tag{1}$$

With the boundary conditions $F(0) = F(H) = 0$ and a given wavenumber k, the wave velocity c in (1) is the eigenvalue of the problem considered. To obtain the upper limit of c (which is of particular importance for analysis of growing perturbations when c is complex and has a positive imaginary part), it is expedient to use an integral equality obtained from (1) by multiplying it by the complex conjugate F^* and integrating over the flow cross-section for the boundary conditions given above. Combining the real and imaginary parts of this integral equation, and introducing ingenious manipulations, Howard (1961) obtained an inequality

$$\left[\left(c_2 - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right] \int \rho[|F'|^2 + k^2|F|^2] + \int \rho N^2 |F|^2 \leqslant 0, \tag{2}$$

where $a = \min U(y)$, $b = \max U(y)$. He dropped the second positive term (thus completely neglecting density stratification) and derived the semicircle theorem. Kochar & Jain (1979) retained the last term in (2), relating it to $I = \int \rho[|F'|^2 + k^2|F|^2]$, and therefore took account of flow stratification. However, in both cases the final result did not contain the wavenumber k, since in all the intermediate stages it entered only into the positive term I, which was supplementary and was dropped in the final inequalities.

The present paper gives the estimate of the second term in inequality (2), following the same pattern as suggested by Kochar & Jain but retaining, where possible, k in its explicit form. For this we use the imaginary part of the integral equation for the function $G(y) = [U(y)-c]^{\frac{1}{2}}F(y)$, which readily follows from (1) (Howard 1961; Kochar & Jain 1979):

$$\int \rho(|G'|^2+k^2|G|^2) = \int \rho(\tfrac{1}{4}U'^2-N^2)\frac{|G|^2}{|U-c|^2}. \tag{3}$$

Taking into account the relation between the functions $G(y)$ and $F(y)$, the following inequality can be written:

$$|G'|^2 \geqslant |U-c||F'|^2+\frac{U'^2|F|^2}{4|U-c|}-|U'||F||F'|. \tag{4}$$

Using this inequality, one can readily obtain from (3) an integral relation

$$(1-4J_0)B^2 \geqslant B^2+E^2+k^2D^2-\int \rho|U'||F||F'|, \tag{5}$$

where

$$B^2 = \int \frac{\rho U'^2|F|^2}{4|U-c|}, \quad E^2 = \int \rho|U-c||F'|^2,$$

$$D^2 = \int \rho|U-c||F|^2.$$

In the analogous inequality obtained by Kochar & Jain the terms E^2 and k^2D^2 are combined in one; as a consequence dependence on the wavenumber vanishes in the final result. In our paper we do not combine the terms, and at this stage the wavenumber k appears in the inequality.

We now transform (5) using the Cauchy–Bunjakowskĭ–Schwartz inequality to obtain the following estimate:

$$\int \rho|U'||F||F'| \leqslant \left[\int \frac{\rho U'^2|F|^2}{|U-c|}\int \rho|U-c||F'|^2\right]^{\frac{1}{2}} = 2BE.$$

Now (5) can be rewritten as

$$E^2-2BE+4J_0B^2+k^2D^2 \leqslant 0. \tag{6}$$

Solving this inequality with respect to E, we obtain

$$B-(B^2-4J_0B^2-k^2D^2)^{\frac{1}{2}} \leqslant E \leqslant B+(B^2-4J_0B^2-k^2D^2)^{\frac{1}{2}}.$$

Since we take interest only in the upper estimate for E, we get

$$E^2+k^2D^2 \leqslant 2B^2\left[1-2J_0+\left(1-4J_0-k^2\frac{D^2}{B^2}\right)^{\frac{1}{2}}\right]. \tag{7}$$

Let us estimate the relation

$$\frac{D^2}{B^2} = \frac{\int \rho|U-c||F|^2}{\int \frac{\rho U'^2|F|^2}{4|U-c|}} \geqslant \frac{c_i\int \rho|F|^2}{\frac{U'^2_{\max}}{4c_i}\int \rho|F|^2} = 4c_i^2(U'_{\max})^{-2}.$$

We should take into account the fact that

$$E^2+k^2D^2 \geqslant c_i \int \rho(|F'|^2+k^2|F|^2),$$

$$B^2 \leqslant \frac{1}{4c_i}\int \rho U'^2|F|^2.$$

With allowance made for these relations, (7) can be rewritten in the form

$$\int \rho U'^2|F|^2 \geqslant \frac{2c_i}{1-2J_0+[1-4J_0-4k^2c_i^2(U'_{max})^{-2}]^{\frac{1}{2}}}\int \rho(|F'|^2+k^2|F|^2). \tag{8}$$

The estimate of the last term on the left-hand side of (2) yields

$$\int \rho N^2|F|^2 \geqslant J_0 \int \rho U'^2|F|^2. \tag{9}$$

Substituting (8) and (9) into (2), we finally obtain

$$\left[\left(c_r-\frac{a+b}{2}\right)^2+c_i^2-\left(\frac{b-a}{2}\right)^2+\frac{2J_0c_i^2}{1-2J_0+[1-4J_0-4k^2c_i^2(U'_{max})^{-2}]^{\frac{1}{2}}}\right] \times \int \rho(|F'|^2+k^2|F|^2) \leqslant 0.$$

Since the integral is definitely positive, the Howard–Kochar–Jain theorem can be generalized as follows:

$$\left(c_r-\frac{a+b}{2}\right)^2+\left[1+\frac{2J_0}{1-2J_0+[1-4J_0-4k^2c_i^2(U'_{max})^{-2}]^{\frac{1}{2}}}\right]c_i^2 \leqslant \left(\frac{b-a}{2}\right)^2. \tag{10}$$

In the complex plane (c_r, c_i) this inequality restricts the range of the wave velocity and the form of the limiting curve explicitly depends both on the minimum Richardson number J_0 and on the wavenumber k of the perturbations. At $k = 0$ the result of Kochar & Jain follows from (10), and the limiting curve takes the form of a semi-ellipse whose minor axis in the direction of c_i depends on J_0. At $J_0 \to 0$ the curve reduces to Howard's semicircle. At the same time, as seen from (10), in the general case $k \neq 0$, $0 < J_0 < 0.25$ the range of c_i according to (10) always lies within Kochar & Jain's semi-ellipse.

3. Estimation of the rate of growth in the presence of external boundaries

One more important relation follows explicitly from (10): stipulation of a non-negative radicand involves the condition

$$k^2c_i^2 \leqslant U'^2_{max}(\tfrac{1}{4}-J_0), \tag{11}$$

which, in fact, sets additional restrictions upon c_i. This condition was first obtained by Howard (1961), but did not explicitly enter into the semicircle theorem. However, it can also be refined taking into account external boundaries of the flow. The inequality (11) is derived from (3), if the term $\int \rho|G'|^2$ is dropped on the left-hand side and the value

$$\frac{1}{c_i^2}U'^2_{max}(\tfrac{1}{4}-J_0) \geqslant \frac{\frac{1}{4}U'^2-N^2}{|U-c|^2}$$

is taken outside the integral sign on the right-hand side of the equation. If the integration is accomplished in finite terms, e.g. between 0 and H, the dropped term may be estimated using a known relation for continuous functions from the space

$L_2(0, H)$ which satisfy zero boundary conditions (Joseph 1976):

$$\int_0^H |G|^2 \, dy \leqslant \frac{H^2}{\pi^2} \int_0^H |G'|^2 \, dy.$$

Taking into account that

$$\frac{\int_0^H \rho |G'|^2 \, dy}{\int_0^H \rho |G|^2 \, dy} \geqslant \frac{\rho_{\min}}{\rho_{\max}} \frac{\int_0^H |G'|^2 \, dy}{\int_0^H |G|^2 \, dy} \geqslant \frac{\rho_{\min} \pi^2}{\rho_{\max} H^2}$$

and

$$\int_0^H \rho \left(\frac{U'^2}{4} - N^2 \right) \frac{|G|^2}{|U-c|^2} dy \leqslant \frac{\max\left(\frac{1}{4}U'^2 - N^2\right)}{c_i^2} \int_0^H \rho |G|^2 \, dy,$$

we obtain from (3)

$$k^2 c_i^2 \leqslant \frac{\max\left(\frac{1}{4}U'^2 - N^2\right)}{\dfrac{\rho_{\min}}{\rho_{\max}} \dfrac{\pi^2}{H^2 k^2} + 1} \leqslant \frac{U'^2_{\max}\left(\frac{1}{4} - J_0\right)}{\dfrac{\rho_{\min}}{\rho_{\max}} \dfrac{\pi^2}{H^2 k^2} + 1}. \tag{12}$$

Assuming that $H \to \infty$, (12) yields the result (11) formerly obtained by Howard. In the general case, however, at finite H (12) gives additional information about the external boundary effect on the rate of growth of shear instability. As seen from (12), with decreasing H the rate of growth decreases in conformity with the numerical computations (Hazel 1972).

4. Further refinement of the results (§2) for specific velocity profiles

The general results obtained in §2 apply to arbitrary density and velocity profiles for a stratified flow. In a number of particular cases, however, parameters of growing perturbations can be refined to a greater extent. To illustrate this, it should be recalled that the apparent inequality

$$0 \geqslant \int (U-a)(U-b) Q = \int U^2 Q - (a+b) \int UQ + ab \int Q \tag{13}$$

plays the central role in proving Howard's theorem. Here $Q = \rho[|F'|^2 + k^2 |F|^2]$. Substituting the integrals (Howard 1961)

$$\int UQ = c_r \int Q, \quad \int U^2 Q = (c_r^2 + c_i^2) \int Q + \int \rho N^2 |F|^2, \tag{14}$$

we obtain (2), which yields Howard's result after dropping the last term on the left-hand side. The closer the right-hand side of (13) is to its upper limit of zero, the more exact is the estimate of the (c_r, c_i) parameter region for growing perturbations. For this purpose, we decrease the absolute value of the integral in (13), substituting Q by a smaller non-negative function

$$\rho[|F'| - k|F|]^2 = Q - 2k\rho |F| |F'|. \tag{15}$$

It gives

$$\begin{aligned} 0 \geqslant \int (U-a)(U-b) \rho [|F'| - k|F|]^2 &= \int (U-a)(U-b) Q \\ -2k \int (U-a)(U-b) \rho |F| |F'| &= \int U^2 Q - (a+b) \int UQ \\ + ab \int Q + 2k \int \left[\left(\frac{b-a}{2} \right)^2 - \left(U - \frac{a+b}{2} \right)^2 \right] &\rho |F| |F'|. \end{aligned} \tag{16}$$

Using (14)

$$0 \geqslant \left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right] \int Q + \int \rho N^2 |F|^2 + 2k \int \left[\left(\frac{b-a}{2}\right)^2 - \left(U - \frac{a+b}{2}\right)^2\right] \rho |F| \, |F'|, \quad (17)$$

we obtain an inequality that generalizes (2) and contains an additional finals term larger than zero. Neglect of the final term only makes the inequality (17) stronger, Howard's and Kochar & Jain's results, as well as those presented in §2, are derived from this inequality in this case.

We now estimate the third term in (17), substituting it either by an equivalent expression or by the one with a smaller absolute value. For this purpose, (5) can be written in the form

$$\int \rho |U'| |F| |F'| \geqslant \int |U-c| Q + \int \frac{\rho N^2 |F|^2}{|U-c|}. \quad (18)$$

Since $|U-c| \geqslant c_i$ and $1/|U-c| \geqslant 1/(b-a)$, (18) may be rewritten in the form

$$\int \rho |U'| |F| |F'| \geqslant c_i \int Q + \frac{1}{b-a} \int \rho N^2 |F|^2. \quad (19)$$

When the condition

$$\int \rho \left[\left(\frac{b-a}{2}\right) - \frac{\left(U - \frac{a+b}{2}\right)^2}{\frac{1}{2}(b-a)}\right] |F| |F'| \geqslant \int h\rho |U'| |F| |F'| \quad (20)$$

is fulfilled, where h is a certain constant with the dimensions of length, the inequality (17) can be strengthened expressing the last component in terms of the integral $\int Q$ with the help of (19). Unfortunately, validity of (20) in the general case has not been proved. However, one can readily distinguish the flow profiles at which relation (20) is fulfilled. In particular, equating the integrands on the left- and right-hand side of (20), we obtain a differential equation for $U(y)$:

$$\frac{dU}{dy} = \pm \frac{2}{h(b-a)} \left[\left(\frac{b-a}{2}\right)^2 - \left(U - \frac{a+b}{2}\right)^2\right]. \quad (21)$$

This equation is readily solved; as a result we obtain

$$U(y) = \pm \left[\frac{a+b}{2} + \left(\frac{b-a}{2}\right) \operatorname{th}\left(\frac{y}{h} - \frac{1}{2}\right)\right]. \quad (22)$$

Here h is used in the sense of the characteristic thickness of a shear layer. This profile is often used in numerical computations of shear-flow stability, since its form provides a good approximation to actually observed flows (Hazel 1972; Turner 1973; Gossard & Hooke 1975). We may state that (20) is fulfilled for the profiles (22) or smoother ones, i.e. profiles for which

$$|U'| < \frac{2}{b-a} \left[\left(\frac{b-a}{2}\right)^2 - \left(U - \frac{b+a}{2}\right)^2\right].$$

For these profiles, (17), with the use of (20), takes the form

$$0 \geqslant \left\{\left(c_r - \frac{a+b}{2}\right)^2 + \left[c_i + \kappa h \left(\frac{b-a}{2}\right)\right]^2 - (1+kh)\left(\frac{b-a}{2}\right)^2\right\} \int Q + (1+kh) \int \rho N^2 |F|^2. \quad (23)$$

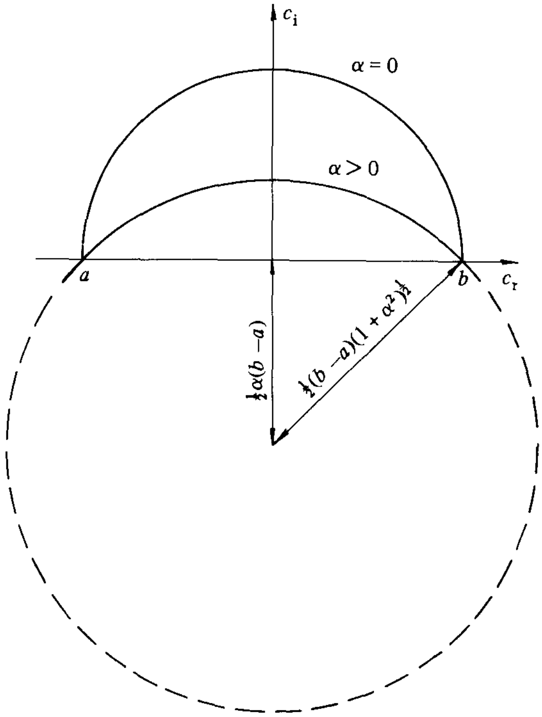

FIGURE 1. Howard's semicircle ($\alpha \equiv kh = 0$) and the segment of the circle ($\alpha > 0$) described by (24).

Dropping the last term in (23), we obtain a generalization of Howard's theorem for the 'most-unstable' case when $N^2 = 0$:

$$\left[c_r - \frac{a+b}{2}\right]^2 + \left[c_i + kh\left(\frac{b-a}{2}\right)\right]^2 \leqslant [1+(kh)^2]\left(\frac{b-a}{2}\right)^2. \tag{24}$$

This implies that at $k = 0$ the range of (c_r, c_i) is restricted to a semicircle with a radius $\frac{1}{2}(b-a)$. At $k \neq 0$ the semicircle transforms to a segment of the circle with a radius $\frac{1}{2}(b-a)[1+(kh)^2]^{\frac{1}{2}}$, but it always rests on the points a and b located on the c_r axis (figure 1). Taking into account the final term, and using its estimate made in §2, we obtain an expression for the boundary of the region (c_r, c_i) which depends on the Richardson number, as well as on the perturbation wavenumber:

$$\left[c_r - \frac{a+b}{2}\right]^2 + \left[c_i + kh\left(\frac{b-a}{2}\right)\right]^2 + \frac{2(1+kh)\,J_0 c_i^2}{1-2J_0+\left[1-4J_0-\dfrac{4k^2c_i^2}{(U'_{max})^2}\right]^{\frac{1}{2}}} \leqslant [1+(kh)^2]\left(\frac{b-a}{2}\right)^2. \tag{25}$$

A more detailed analysis of this inequality is given in §5. Here we can note that the range of (c_r, c_i) defined by (25) is confined not only to Howard's semicircle and Kochar & Jain's semi-ellipse but also to the region defined by (10).

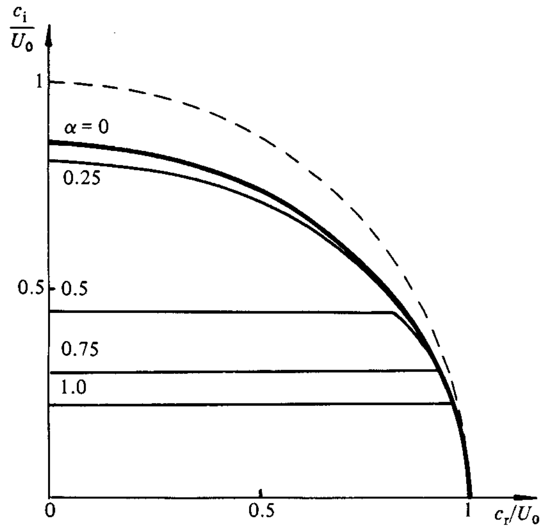

FIGURE 2. Dependence of the complex wave-velocity range for growing perturbations on the dimensionless wavenumber α, plotted using (26) at $J_0 = 0.2$ (the curves are symmetrical with respect to the c_i axis). The dashed line shows Howard's semicircle, and the heavy curve is given for Kochar & Jain's semi-ellipse.

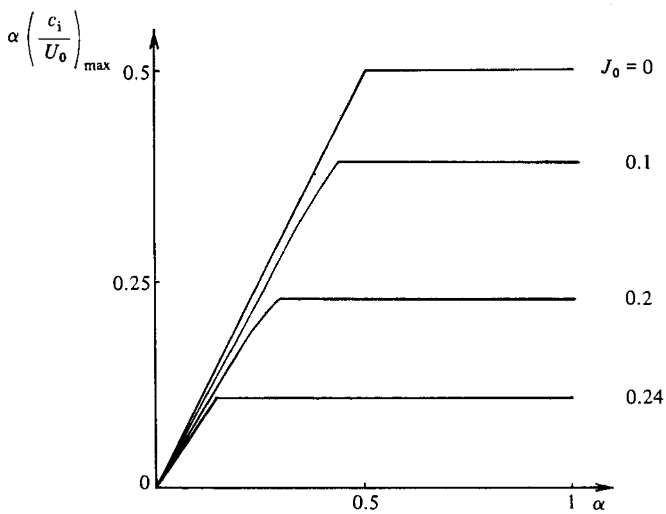

FIGURE 3. Estimates of the maximum growth rate αc_i versus the wavenumber α at different J_0.

5. Discussion

To analyse the obtained results, consider some smooth symmetric profiles for which $a = -b \equiv -U_0$. Introducing dimensionless variables $\xi = y/h$, $\alpha = kh$, we rewrite (10) in the form

$$c_r^2 + \left[1 + \frac{2J_0}{1 - 2J_0 + \left[1 - 4J_0 - 4\alpha^2 c_i^2 \left(\frac{dU}{d\xi}\right)_{max}^{-2}\right]^{\frac{1}{2}}}\right] c_i^2 \leqslant U_0^2. \tag{26}$$

Figure 2 gives an example of the curves defining the range of c at different α and $J_0 = 0.2$. The breaks of the curves indicate that the radicand in (26) becomes negative

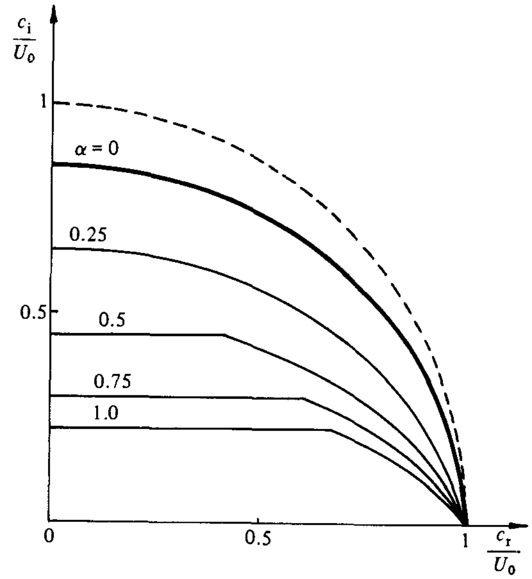

FIGURE 4. Range of c as a function of α for the velocity profile of the type (22) plotted using (27) at $J_0 = 0.2$. The dashed line is given for Howard's semicircle, the heavy line is for Kochar & Jain's semi-ellipse.

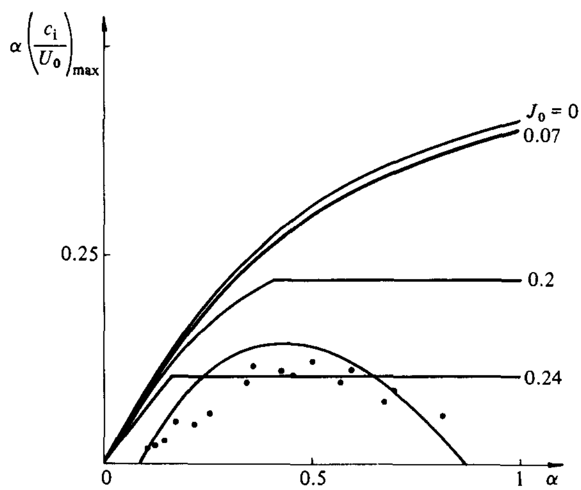

FIGURE 5. Estimates of the maximum growth rate versus the wavenumber for the velocity profile of the type (22) at different J_0. The dots are given for the experimental data (Scotti & Corcos 1969). The smooth curve in their vicinity presents the computed shear flow with a similar velocity profile (Hazel 1972).

at sufficiently large α. Therefore the additional restriction stipulated by (11) occurs in the imaginary part of the wave velocity. The heavy line is Kochar & Jain's semi-ellipse at $\alpha = 0$, the dashed line is for Howard's semicircle. As seem from figure 2, the maximum of the imaginary part of the wave velocity depends on the dimensionless wavenumber as well as on J_0. Dependence of the maximum growth rate on the wavenumber is of great importance for practical calculations and applications. The inequalities (10) and (11) enable us to estimate the dependence of the maximum growth rate on α. The family of broken lines in figure 3 represent the estimates of

αc_i versus α at different J_0. The growth rate in the region of small α increases with α, reaches its maximum and remains constant. Unfortunately, the expressions obtained do not allow us to explain the decrease in the growth rate after reaching its maximum that usually occurs in experiments and numerical computations of instability in smooth profiles (Drazin & Howard 1966; Turner 1973; Gossard & Hooke 1975).

We now consider a velocity profile of the type (22), assuming that density is an arbitrary smooth function of the y-coordinate. We rewrite (25) in dimensionless variables

$$c_r^2+(c_i+\alpha U_0)^2+\frac{2J_0(1+\alpha)\,c_i^2}{1-2J_0+[1-4J_0-4\alpha^2 c_i^2\, U_0^{-2}]^{\frac{1}{2}}}\leqslant (1+\alpha^2)\,U_0^2. \tag{27}$$

The range of (c_r, c_i) defined by (27) is shown in figure 4 for different values of α and $J_0 = 0.2$. As seen in figure 4, breaks of the curves stipulated by (11) are also available here, but they appear at α larger than in (26). It is easily seen that the curves corresponding to equal values of α are located in figure 4 lower than in figure 2. Figure 5 shows estimates of the growth rate as a function of the wavenumber for the example under consideration at different J_0. A similar velocity profile was observed in laboratory experiments (Scotti & Corcos 1969) and used in numerical computations (Hazel 1972). The dots in figure 5 mark the experimental data obtained by Scotti & Corcos at $J_0 = 0.07$; the smooth curve in their vicinity was computed by Hazel. The estimate for this case described by (27) is given by a heavy line. Here we could not explain the decrease in the growth rate at large α either. Within the frames of the obtained formulae, all the curves at $\alpha \to \infty$ tend to their asymptotic values $\alpha c_i/U_0 \leqslant 0.25 - J_0$ defined by (11). The comparison showed that the maximum growth rate estimated by (11) is about three times larger than that obtained in the numerical computations for different smooth profiles (Miles & Howard 1964; Hazel 1972).

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